Natural Frequencies of a Tapered Cantilever Beam of Constant Thickness and Linearly Tapered Width

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A method for determination of natural frequencies of a tapered cantilever beam in free bending vibration by a rigid multibody system is proposed. The considerations are performed in the frame of Euler-Bernoulli beam theory. The method consists of two steps. In the first step, the tapered cantilever beam is approximated by n flexible straight beam, and after that all of the n segments are divided into k segments. In the second step, all of the flexible straight beams are replaced by three rigid beams connected through revolute and prismatic joints with the corresponding springs in them. The results of the proposed method are compared with similar methods proposed in literature.

Keywords: Free bending vibration, tapered cantilever beam, rigid multibody system, natural frequencies

1. INTRODUCTION

Research on dynamic characteristics of flexible tapered cantilever beams is very important in different engineering fields. These types of beams appear most frequently as the result of a need for saving in material, reduction of weight, better utilization of material, increased rigidity, etc. A significant number of papers dedicated to the solution of this problem have been recently published.

This paper presents a new approach to approximative determination of natural frequencies of free vibration of this type of beams using the main ideas presented in [6] and [7]. A short analysis and adaptation of approaches from [1], [2], and [3] will be carried out for the purpose of comparing the obtained results with the results from similar approaches so that the procedure of analysis of this type of beams could be feasible. Comparison between the presented approach and the two mentioned approaches will be performed on the example from [4], where exact values of frequencies of the stepped cantilever beam are determined. Also, the results of the presented approach will be compared with the results from [5], where the tapered cantilever beam of a rectangular cross section, constant thickness and linearly tapered width is analysed.

2. A RIGID MULTIBODY MODEL OF A TAPERED CANTILEVER BEAM

Let us consider free vibration of the flexible tapered cantilever beam with the length L, where its end A is clamped, and the end B is free, as shown in Fig. 1. The beam thickness h is constant, whereas its width changes linearly along the beam, starting from b_A in the clamped end of the beam, up to b_B at the free end:

$$f_b(x) = \frac{b_B - b_A}{L} x + b_A,\ 0 \le x < L, \tag{1}$$

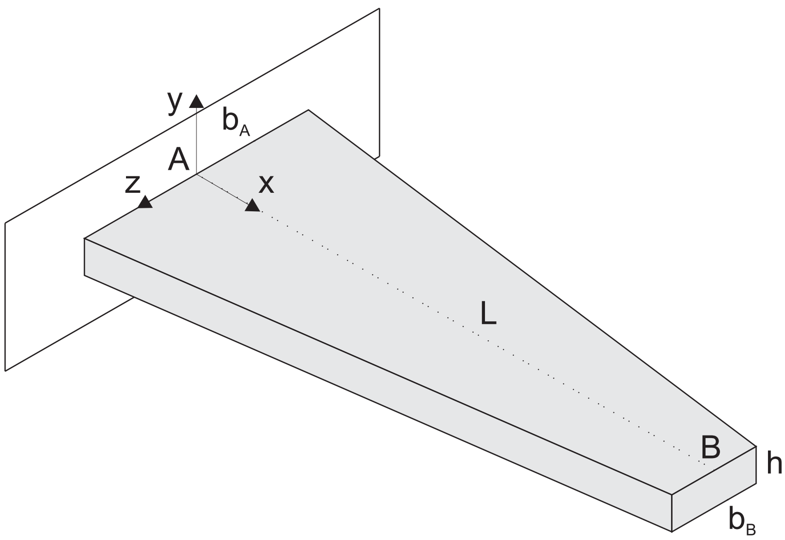

Figure 1: Tapered cantilever beam

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A rigid multibody model of the tapered cantilever beam will be created in two steps. In the first step, the exact shape of the cantilever beam is approximated with *n* flexible segments of constant width. In the second step, each n flexible segment is divided into *k* equal flexible segments. In further text, the division in the first step will be called primary division, and the division in the second step will be called secondary division. Primary division should approximate the exact shape of the cantilever beam in the best way, and secondary division should additionally increase accuracy.

The parameters which define each of the obtained segments are:

-the modulus of elasticity of the material E,

-the shear modulus of the material:

$$G=\frac{E}{2(1+\mu)}, \tag{2}$$

-the Poisson coefficient μ,

-the density of the material ρ,

-the length of the segment after primary division:

$$L_i=\frac{L}{n}, i=1,...,n, \tag{3}$$

-the width of the segment after primary division:

$$b_i=\begin{cases} f_b(\frac{L_1}{2}),\ i=1, \\ f_b(\sum_{k=1}^{i-1} L_k+\frac{L_i}{2}),\ 1<i\le n, \end{cases} \tag{4}$$

-the area of the cross section of the segment after primary division A_i,

-the axial moment of inertia for the principal axis z of the cross section of the beam after primary division I_{zi},

The approximative shape of the tapered beam after primary and secondary divisions is shown in Fig. 2 by dashed lines.

2.1. Our approach

Each of $n{\cdot}k$ flexible segments is divided into three rigid segments, where the first and second rigid segments are interconnected through a prismatic joint, and the second and third segments through a revolute joint (see Fig. 3a).

The springs of corresponding stiffness are placed in the joints. The approximative model of the flexible tapered cantilever beam is thus obtained in the form of an opened kinematic chain without branching made of $2n{\cdot}k$ rigid segments connected through the corresponding joints and springs in them (see Fig. 4). Let us determine the parameters of the observed mechanical system which are necessary for further considerations.

The stiffness of springs in the joints of the *i*-th segment based on [7], for the case of bending of the beam in one plane, are:

$$c_r=k^3\frac{12E\cdot I_{zi}}{L_i^3},\ c_s=k\frac{EI_{zi}}{L_i}, \tag{5}$$

where the indices r and s are:

$$\begin{aligned} &r=2j-1+2k(i-1), \\ &s=2j+2k(i-1),\ i=\overline{1,n},\ j=\overline{1,k}, \end{aligned} \tag{6}$$

The length of the rigid segments is:

$$l_r=\frac{l_i'}{2}, \tag{7}$$

$$l_s=\begin{cases} \frac{3}{2}l_i',\ j<k, \\ l_i'+l_i'',\ j=k\wedge i<n, \\ l_i',\ j=k\wedge i=n, \end{cases} \tag{8}$$

where

$$l_i'=\frac{L_i}{2k},\ l_i''=\frac{L_{i+1}}{4k}, \tag{9}$$

The mass of the rigid segments is:

$$\begin{aligned} &m_r=\rho A_i l_r, \\ &m_s=\begin{cases} \rho\left(A_i l_i'+A_{i+1}l_i''\right),\ j=k\wedge i<n, \\ \rho A_i l_s, \end{cases} \end{aligned} \tag{10}$$

The position of the centre of mass of each rigid segment is defined by the local position vector of the centre of mass in relation to the beginning of the segment:

$$\boldsymbol{\rho}_{c_u}=\begin{bmatrix}\xi_{c_u} & \eta_{c_u} & \zeta_{c_u}\end{bmatrix}^T, \tag{11}$$

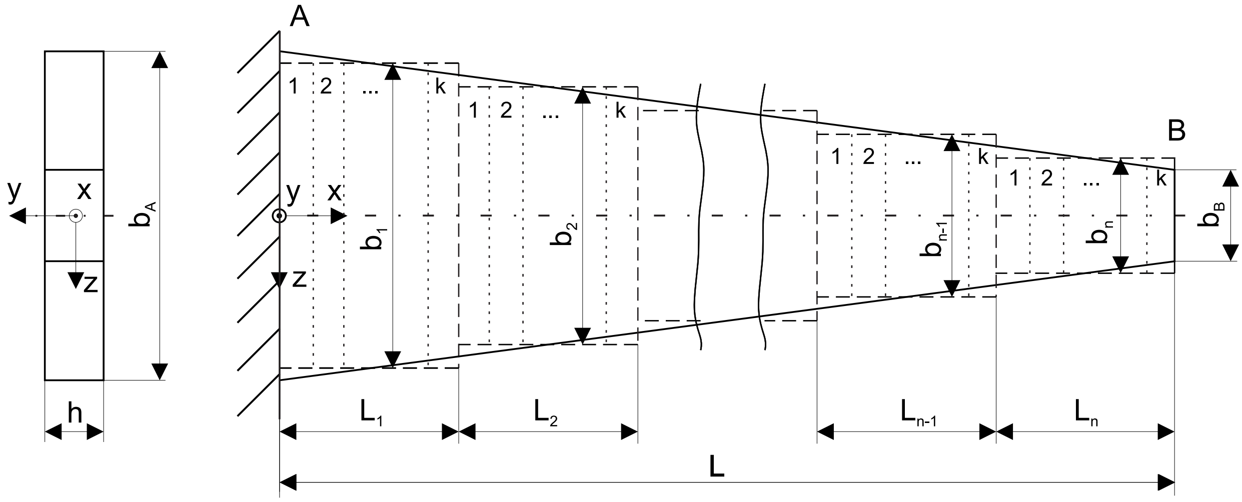

Figure 2: An approximation of the cantilever tapered beam by stepped beams

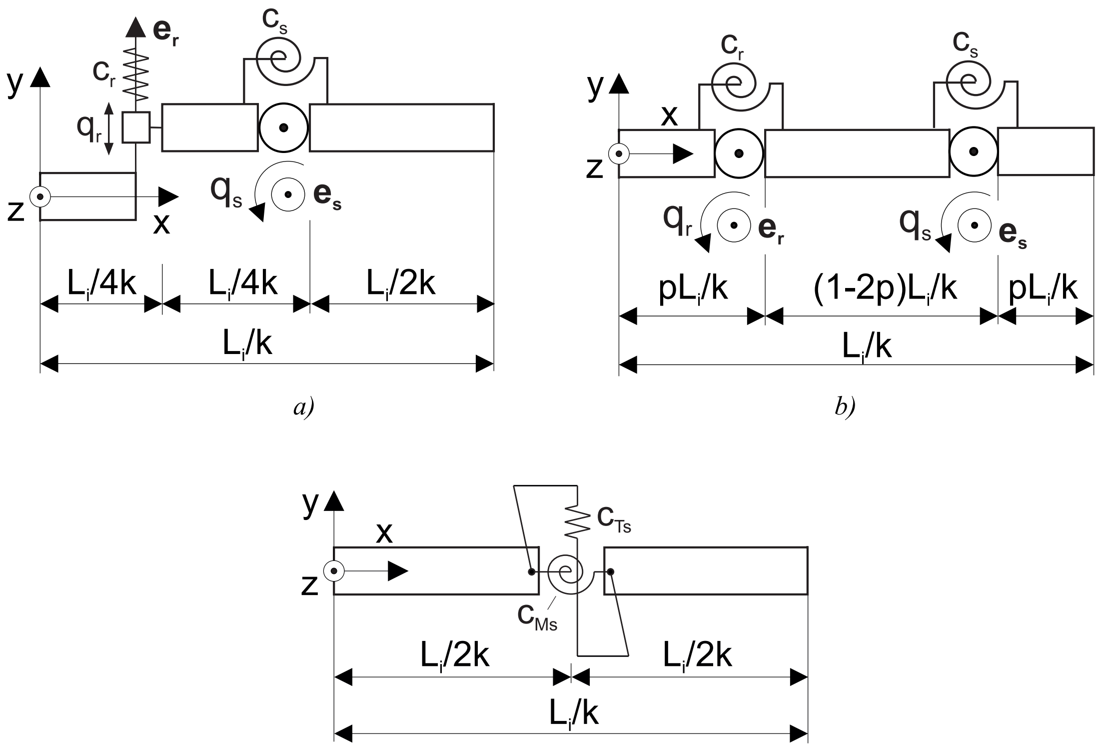

Figure 3: The rigid multibody model of the i-th flexible beam segment: a) Presented approach, b) Ref. [1], c) Ref. [2], [3]

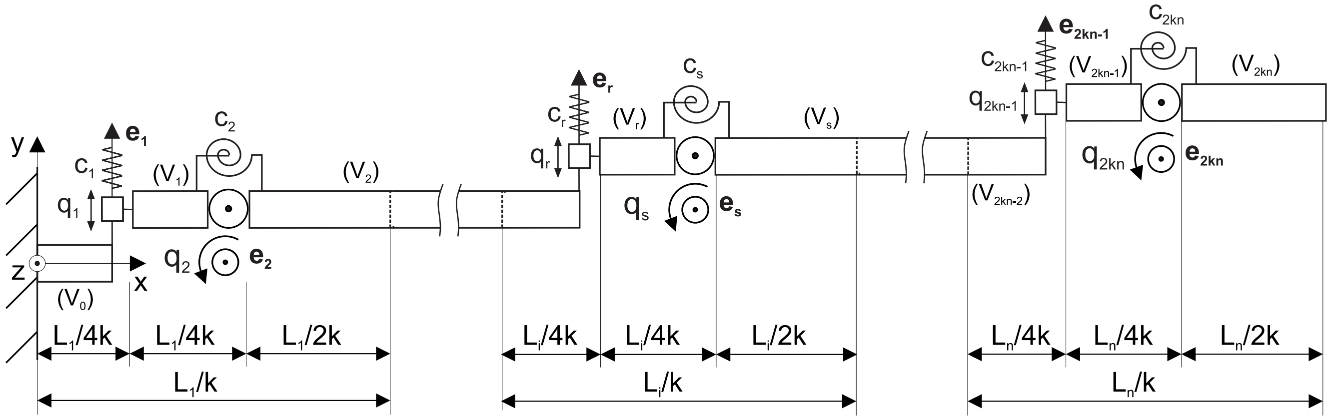

Figure 4: The rigid multibody model of the flexible beam

where

$$\xi_{c_r} = 0.5 \cdot l_r,$$

$$\xi_{c_s} = \begin{cases} \dfrac{0.5 A_i l_i'^2 + A_{i+1} l_i'' \left(l_i' + 0.5 \cdot l_i'' \right)}{A_i \cdot l_i' + A_{i+1} l_i''}, & j = k \wedge i < n, \\ 0.5 \cdot l_s, & \end{cases}$$

$$\eta_{c_u} = 0,\ \zeta_{c_u} = 0,\ u = \overline{1,2kn}, \qquad (12)$$

The local vectors of the rigid segments are:

$$\boldsymbol{\rho}_u = \begin{bmatrix} l_u & 0 & 0 \end{bmatrix}^T, u = \overline{1,2kn}, \qquad (13)$$

The moment of inertia of the rigid segment in relation to the axis ζ perpendicular to the plane of rotation is:

$$J_{c_r \zeta} = \frac{m_r}{12}\left(a_r^2 + l_r^2\right),$$

$$J_{c_s \zeta} = \begin{cases} \rho A_i l_i' \left(\dfrac{1}{12}\left(a_i^2 + l_i'^2\right) + \left(\xi_{c_s} - \dfrac{l_i'}{2}\right)^2 \right) \\ + \rho A_{i+1} l_i'' \left(\dfrac{1}{12}\left(a_{i+1}^2 + l_i''^2\right) + \left(l_i' + \dfrac{l_i''}{2} - \xi_{c_s}\right)^2 \right), \\ j = k \wedge i < n, \\ \dfrac{m_s}{12}\left(a_i^2 + l_r^2\right), \end{cases} \qquad (14)$$

where:

$$\mathbf{a}_i = \begin{cases} h, \text{ if the cross section is rectangular },\\ \frac{\sqrt{3}}{2} d_i, \text{ if the cross section is circular }, \end{cases} \tag{15}$$

The unit vectors of the axis of the u-th joint are:

$$\mathbf{e}_u = \begin{cases} [0 \quad 1 \quad 0]^T, \text{ if the } u\text{-th joint is prizmatic},\\ [0 \quad 0 \quad 1]^T, \text{ if the } u\text{-th joint is revolute}, \end{cases} \tag{16}$$

where $u = \overline{1, 2kn}$.

The coefficients χ_u and $\bar{\chi}_u$ represent identifiers of the joint type, where it holds that:

$$\chi_u = \begin{cases} 1, \text{ if the } u\text{-th joint is prismatic},\\ 0, \text{ if the } u\text{-th joint is revolute}, \end{cases} \tag{17}$$

as well as that $\bar{\chi}_u = 1 - \chi_u$

2.2. Approach from [1]

Each of $n{\cdot}k$ flexible segments is divided into three rigid segments, where the first and second rigid segments, as well as the third and fourth ones, are interconnected with a revolute joint (see Fig. 3b). The springs of the corresponding rigidity are placed in those joints. Similarly to our approach, the approximative model of the flexible tapered cantilever beam in the form of an open kinematic chain without branching made of $2n{\cdot}k$ rigid segments connected with the corresponding joints and springs in them is obtained. Let us determine the parameters of the observed mechanical system which are necessary for further considerations.

The stiffnesses of springs in the joints of the i-th segment based on [1] are:

$$c_r = c_s = 2k\frac{EI_{zi}}{L_i},\ i = \overline{1, n}, \tag{18}$$

where the indices r and s are defined in the expression (6).
The length of the rigid segments is:

$$l_r = \frac{1-2p}{p} l_i', \tag{19}$$

$$l_s = \begin{cases} 2l_i', j < k,\\ l_i' + l_i'', j = k \wedge i < n,\\ l_i',\ j = k \wedge i = n, \end{cases} \tag{20}$$

where

$$l_i' = p\frac{L_i}{k},\ l_i'' = p\frac{L_{i+1}}{k}, \tag{21}$$

and where $p = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)$ is the coefficient of division of the beam. Reference [1] shows that, especially for this value of the coefficient, the assumed model of the beam is reduced to a simpler shape which contains springs only in the joints (see Fig. 3b). In an opposite case, the model of the beam also contains a spring which connects the first and third rigid bodies, and then the process of modelling the flexible beam by this method becomes considerably complicated.

The local position vector of the centre of mass of rigid segments $\boldsymbol{\rho}_{c_u}$, the mass of the rigid segments m_u and the moment of inertia of the rigid segment in relation to the axis ζ $J_{c_u\zeta}$ ($u = 1, \overline{2k \cdot n}$) may be defined from the expressions (10)-(14), where l_i' i l_i'' are given in the expression (21).
The unit vectors of the axis of the u-th joint are:

$$\mathbf{e}_u = [0 \quad 0 \quad 1]^T. \tag{22}$$

All joints in the kinematic chain are revolute, so that:

$$\bar{\chi}_u = 1,\ u = \overline{1, 2n \cdot k} \tag{23}$$

2.3. Approach from [2], [3]

References [2] and [3] propose discretization of each of $n{\cdot}k$ flexible segments so that they are divided into two equal rigid segments which are interconnected by one cylindrical spring and one revolute spring with the corresponding rigidity (see Fig. 3c). This division results in an open kinematic chain without branching made of $n{\cdot}k$ rigid segments connected by the corresponding springs.
The stiffnesses of springs in the joints of the u-th segment based on [2] and [3] are:

$$c_{T_s} = k\frac{GA_i}{L_i},\ c_{M_s} = \frac{EI_{zi}}{L_i}, \tag{24}$$

where the index s is:

$$s = j + k(i-1),\ i = \overline{1, n}, j = \overline{1, k}, \tag{25}$$

The length of the rigid segments is:

$$l_s = \begin{cases} 2l_i', j < k,\\ l_i' + l_i'', j = k \wedge i < n,\\ l_i',\ j = k \wedge i = n, \end{cases} \tag{26}$$

where

$$l_i' = \frac{L_i}{2k},\ l_i'' = \frac{L_{i+1}}{2k}, \tag{27}$$

The local position vector of the centre of mass of the rigid segments $\boldsymbol{\rho}_{c_u}$, the local vectors of the rigid segments $\boldsymbol{\rho}_u$, the mass of the rigid segments m_u and the moment of inertia of the rigid segment of constant width in relation to the axis ζ $J_{c_u\zeta}$ ($u = 1, \overline{2k \cdot n}$) may be determined from the expressions (10)-(14), where l_i', l_i'' i l_s are given in (26) and (27).

3. EIGENVALUE PROBLEM

Reference [1] and our approach use relative coordinates for description of the system, whereas [2] and [3] use absolute coordinates. That is the reason why the formation of differential equations of motion will be presented for the cases of using relative coordinates (for our approach and the approach in [1]) and absolute coordinates (for the approaches in [2] and [3]).

3.1. Relative coordinates

The potential energy of the system of springs in the joints reads:

$$\Pi_c = \frac{1}{2}\sum_{u=1}^{2kn} c_u q_u^2, \tag{28}$$

where q_u (u=1,…,2kn) are relative joint displacements.
The kinetic energy of the system is

$$T = \frac{1}{2}\sum_{\alpha=1}^{2kn}\sum_{\beta=1}^{2kn} m_{\alpha\beta}(\mathbf{q})\dot{q}_\alpha \dot{q}_\beta, \tag{29}$$

where an overdot denotes the derivative with respect to time, $\mathbf{q}=[q_1, q_2, \dots, q_{2kn}]^T$ is the vector of generalized coordinates and

$$m_{\alpha\beta}(\mathbf{q}) = \sum_{u=\beta}^{2kn}\left(m_u \frac{\partial \mathbf{r}_{c_u}}{\partial q_\alpha}^T \frac{\partial \mathbf{r}_{c_u}}{\partial q_\beta} + \bar{\chi}_\alpha \bar{\chi}_\beta J_{C_u\zeta} \mathbf{e}_\alpha{}^T \mathbf{e}_\beta \right), \tag{30}$$

the metric tensor coefficient of the inertia matrix of the system. For more details see [9].

In Equation (30), m_u is the mass of the u-th rigid segment in the chain, $J_{C_u\zeta}$ is its axial moment of inertia relative to the principal axis which is perpendicular to the plane of beam bending, $\mathbf{r}_{cu}$ is the vector of position of the centre of masses of the rigid body (V_u) in relation to the inertial frame $Axyz$. The configuration $\mathbf{q}_0=[q_1=0, \dots, q_{2kn}=0]^T$ in which $\dot{q}_u(t) \equiv 0$, $\ddot{q}_u(t) \equiv 0$ (u=1,...,2kn) corresponds to the equilibrium position of the flexible beam shown in Fig. 4 in the absence of gravity and force at the free end of the beam B. Linearized differential equations of motion of the considered system of rigid bodies in the surroundings of the equilibrium position read (see [8]):

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}_{2n\times 1}, \tag{31}$$

where $\mathbf{0}_{2n\ 1} \in R^{2kn\times 1}$, $\mathbf{K} = diag(c_1, \dots, c_{2kn})$ is the stiffness matrix, and $\mathbf{M} \in R^{2kn\times 2kn}$ is the mass matrix, whose members are:

$$m_{\alpha\beta}(\mathbf{q}_0) = \sum_{u=\beta}^{2kn} m_i \left(\frac{\partial \mathbf{r}_{c_u}}{\partial q_\alpha}\right)_{\mathbf{q}_0}^T \left(\frac{\partial \mathbf{r}_{c_u}}{\partial q_\beta}\right)_{\mathbf{q}_0} + \sum_{u=\beta}^{2kn} \bar{\chi}_\alpha \bar{\chi}_\beta J_{C_u\zeta} \mathbf{e}_\alpha(\mathbf{q}_0)^T \mathbf{e}_\beta(\mathbf{q}_0), \ (\alpha,\beta = \overline{1,2kn}), \tag{32}$$

The partial derivative of the position vector $\mathbf{r}_{cu}$ relative to generalized coordinate q_α at the position $\mathbf{q}_0$ reads:

$$\left(\frac{\partial \mathbf{r}_{c_u}}{\partial q_\alpha}\right)_{\mathbf{q}_0} = \begin{cases} \bar{\chi}_\alpha \mathbf{e}_\alpha(\mathbf{q}_0) \times \left(\sum_{k=\alpha+1}^{u} \boldsymbol{\rho}_{k-1}(\mathbf{q}_0) + \boldsymbol{\rho}_{C_u}(\mathbf{q}_0) \right) \\ + \chi_\alpha \mathbf{e}_\alpha(\mathbf{q}_0), \quad \alpha < u, \\ \bar{\chi}_\alpha \mathbf{e}_\alpha(\mathbf{q}_0) \times \boldsymbol{\rho}_{C_u}(\mathbf{q}_0) + \chi_\alpha \mathbf{e}_\alpha(\mathbf{q}_0), \quad \alpha = u, \\ \mathbf{0}, \quad \alpha > u \end{cases} \tag{33}$$

3.2. Absolute coordinates

The potential energy of the system of springs in the joints reads:

$$\Pi_c = \frac{1}{2}\sum_{v=1}^{kn}\left(c_{M_v} \Delta\varphi_v^2 + c_{T_v} \Delta y_v^2 \right), \tag{34}$$

where $\Delta\varphi_v$ and Δy_v are relative joint displacements which, expressed as a function of absolute coordinates, read:

$$\Delta\varphi_v = \varphi_v - \varphi_{v-1}, \tag{35}$$

$$\Delta y_v = y_v - zl_v \varphi_v - y_{v-1} - zr_{v-1}\varphi_{v-1}, \ \varphi_0 = 0, \ y_0 = 0. \tag{36}$$

$$zl_v = \xi_{c_v}, \ zr_v = l_v - \xi_{c_v}, \tag{37}$$

The absolute coordinates y_v and φ_v represent transverse displacements of the centres of masses and rotation about that centres of the v-th rigid segment in relation to the horizontal position, respectively. Axial displacements of the centres of masses of the v-th rigid segment are neglected because of the assumption of small deformations of the beam.
The kinetic energy of the system is

$$T = \frac{1}{2}\sum_{v=1}^{kn}\left(m_v \dot{y}_v{}^2 + J_{c_v\zeta} \dot{\varphi}_v{}^2 \right), \tag{38}$$

where an overdot denotes the derivative with respect to time. By applying the Lagrange equations of the second kind for the case of conservative systems,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_v}\right) - \frac{\partial L}{\partial y_v} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_v}\right) - \frac{\partial L}{\partial \varphi_v} = 0, \tag{39}$$

where $v = \overline{1, k\cdot n}$, and $L = T - \Pi$ is the Lagrange function, differential equations of motion of the mechanical system are obtained in the form:

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{0}_{2kn\times 1}, \tag{40}$$

where $\mathbf{0}_{2kn\ 1} \in R^{2kn\times 1}$, $\mathbf{z}=[\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_{kn}]^T$ is the vector of absolute coordinates, and it holds that $\mathbf{z}_v=[y_v, \varphi_v]^T$ (v=1,…,kn).
The mass matrix is:

$$\mathbf{M} = diag(\mathbf{M}_{1,1}, \mathbf{M}_{2,2}, \dots, \mathbf{M}_{kn,kn}), \tag{41}$$

where:

$$\mathbf{M}_{v,v} = diag\left(m_v, J_{c_v\zeta}\right), \tag{42}$$

The stiffness matrix is:

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{1,1} & .. & 0 & 0 & 0 & .. & 0 \\ .. & .. & : & : & : & : & : \\ 0 & .. & \mathbf{K}_{v-1,v-1} & \mathbf{K}_{v-1,v} & 0 & .. & 0 \\ 0 & .. & \mathbf{K}_{v,v-1} & \mathbf{K}_{v,v} & \mathbf{K}_{v,v+1} & .. & 0 \\ 0 & .. & 0 & \mathbf{K}_{v+1,v} & \mathbf{K}_{v+1,v+1} & .. & 0 \\ : & :: & : & : & : & :: & : \\ 0 & .. & 0 & 0 & 0 & .. & \mathbf{K}_{kn,kn} \end{bmatrix}, \tag{43}$$

where:

$$\mathbf{K}_{v,v-1} = \begin{bmatrix} -c_{T_{v-1}} & -c_{T_{v-1}} zr_{v-1} \\ c_{T_{v-1}} zl_{v-1} & -c_{M_v} + c_{T_{v-1}} zr_{v-1} zl_v \end{bmatrix}, \tag{44}$$

$$\mathbf{K}_{v,v} = \begin{bmatrix} c_{T_{v-1}} + c_{T_v} & c_{T_{v-1}} zl_v + c_{T_v} zr_v \\ -c_{T_{v-1}} zl_v + c_{T_v} zr_v & c_{M_{v-1}} + c_{M_v} + c_{T_{v-1}} zl_v{}^2 + c_{T_v} zr_v{}^2 \end{bmatrix}, \tag{45}$$

$$\mathbf{K}_{v,v+1} = \begin{bmatrix} -c_{T_v} & c_{T_v} zl_{v+1} \\ -c_{T_v} zr_v & -c_{M_v} + c_{T_v} zl_{v+1} zr_v \end{bmatrix}, \tag{46}$$

For more details, see [2] or [3].

Finally, the eigenvalue problem, formulated on the basis of (31) and (40), reads:

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right)\mathbf{v} = \mathbf{0}_{2kn\times 1}, \quad (47)$$

where ω is the natural frequency of free vibration of the flexible tapered cantilever beam, and $\mathbf{v} \in R^{2kn\ 1}$ represents the eigenvector which corresponds to the given frequency. Approximate values of natural frequencies of the considered cantilever beam are obtained by solving the eigenvalue problem (47).

4. NUMERICAL EXAMPLE AND VERIFICATION OF THE METHOD

Verification of the efficiency of the presented method will be performed through two examples. The first example will treat the problem of determination of natural frequencies of the flexible cantilever beam with three stepped changes of the circular cross section. Thus, primary division is carried out in advance, so that *n*=4, and the influence of secondary divisions of the beam on the accuracy of our method will be analyzed. Exact values of natural frequencies of such a beam are determined in [4], so it is a good example for comparing the accuracy of the proposed approach with the relevant approaches presented in [1] and [2]. The second example analyzes the tapered cantilever beam of a rectangular cross section, constant thickness and linearly variable width. The influence of primary division on the accuracy of our method will be analyzed in this example. The results achieved by using our approach will be compared with the results from [5].

4.1. Example 1

Let us observe the flexible cantilever beam with three stepped changes of the circular cross section with the following characteristics:

- Young's modulus: $E = 2.068\times 10^{11}\ N/m^2$,
- mass density: $\rho = 7850\ kg/m^3$,
- total length: $L = 2.0\ m$,
- diameter: $d_1 = 0.03\,m$,
- diameters ratio: $d_2/d_1 = 0.8$ $d_3/d_1 = 0.65$, $d_4/d_1 = 0.25$,
- length of the segments: $L_1 = 0.25L$, $L_2 = 0.3L$, $L_3 = 0.25L$, $L_4 = 0.2L$,
- area of the cross section of the segment after primary division of the beam: $$A_u = \frac{\pi d_u^2}{4},$$
- axial moment of inertia for the principal axis z of the cross section of the beam: $$I_{zu} = \frac{\pi d_u^4}{64},$$

In further considerations, for convenience of comparisons with the results from paper [4], the non-dimensional frequency coefficients $\beta L = \sqrt[4]{\omega^2 \rho A_1 L^4/(EI_{1z})}$ are used. Using the above theory, the approximative numerical values of the first three non-dimensional frequency coefficients are obtained. These frequency coefficients along with the corresponding relative errors are shown in Table 1. The errors are calculated as:

$$\frac{approximative\ \ value}{exact\ \ value}\cdot 100 - 100[\%].$$

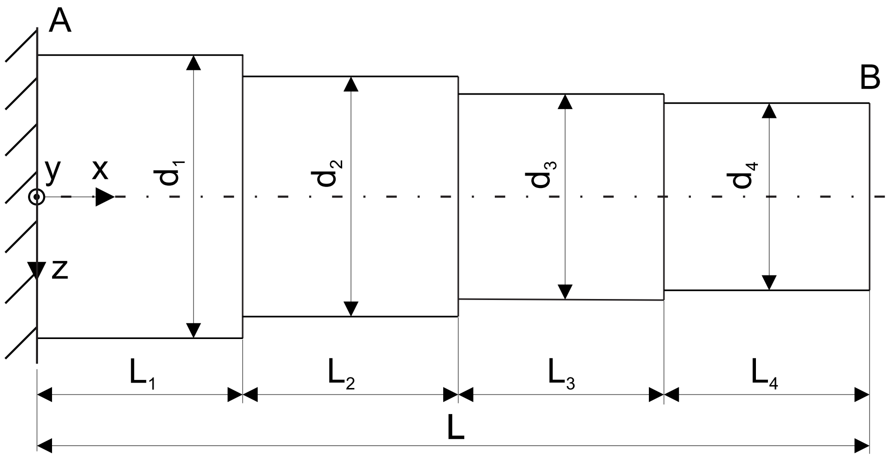

Figure 5: The three-stepped cantilever beam

Table 1 gives the comparative results obtained by using all three presented methods of discretization depending on the number of secondary divisions of each segment of the beam. It also shows relative errors of the obtained values of frequencies in relation to the exact values of frequencies from [4]. It can be noticed that the values of obtained frequencies, at the increased number of secondary divisions of beam segments, converge faster toward the exact values if our approach is used, exept for the third frequency where the approach from [2] is slightly better. Besides, the relative error in determination of the first frequency with one division of the beam segment is 0.059 %, i.e. the error is far smaller than 1%. This fact is particularly important if it is taken into account that the values of the first frequency are of most significance in studying dynamic characteristics of various technical objects.

4.2. Example 2

Let the tapered cantilever beam of constant thickness and linearly tapered width be given (see Fig. 1). The material of the beam is the same as in the previous example. The beam length is $L = 0.5\ m$, and the thickness is $h = 0.005\ m$.

The area of the cross section of the segment after primary division of the beam is:

$$A_u = b_u h,$$

The axial moment of inertia for the principal axis z of the cross section of the beam is:

$$I_{zu} = \frac{b_u h^3}{12},$$

The beam width at the beginning and the end of the beam will be varied in order to obtain necessary relations of these dimensions for the needs of comparison of results. That is why the parameter related to the degree of beam tapering is introduced:

$$c = 1 - \frac{b_B}{b_A}, \tag{48}$$

Let us also introduce the concept of the *i*-th non-dimensional frequency $\bar{\omega}_i$, which is connected with the *i*-th frequency $\omega_i\,(rad/s)$ in the following way:

$$\bar{\omega}_i = \sqrt{\frac{\rho A_1 L^4}{E \cdot I_{zi}}} \cdot \omega_i, \tag{49}$$

Table 2 gives the values of the first three non-dimensional frequencies, where the value of the parameter *c* changes from 0 to 1, with the step 0.1, and for *n*=10 and *n*=20, respectively. It can be noticed that there is very good agreement between our results and the results from [5]. It is obvious that the convergence of frequency toward the values from [5] is faster at smaller values of the parameter *c*, i.e. when the beam is less tapered (see Fig. 6 and Fig. 7). In that case it is enough for the number of segments of constant width (primary divisions) to be n=10, and to achieve the satisfactory accuracy.

Table 1: Natural frequencies of the cantilever beam – comparison of the present paper results and the results from [4]

Number of divisions	Non-dimensional frequency coefficients								
	$\beta_1 L$			$\beta_2 L$			$\beta_3 L$		
	Relative error [%]			Relative error [%]			Relative error [%]		
	Our appr.	Appr. from ref. [1]	Appr. from ref. [2]	Our appr.	Appr. from ref. [1]	Appr. from ref. [2]	Our appr.	Appr. from ref. [1]	Appr. from ref. [2]
1	2.51159	2.51000	2.56814	4.31315	4.43415	4.87846	5.47403	5.79999	6.10351
	0.059	-0.004	2.312	-2.975	-0.254	9.741	-5.938	-0.337	4.878
2	2.51200	2.49390	2.52396	4.43100	4.43826	4.53171	5.74267	5.77220	5.86927
	0.076	-0.645	0.552	-0.324	-0.161	1.941	-1.322	-0.815	0.853
3	2.51114	2.49524	2.51612	4.44160	4.43855	4.48271	5.78781	5.77304	5.84107
	0.042	-0.592	0.240	-0.086	-0.155	0.839	-0.546	-0.800	0.369
5	2.51051	2.49909	2.51213	4.44470	4.44009	4.45832	5.80856	5.78317	5.82606
	0.016	-0.439	0.081	-0.016	-0.120	0.290	-0.190	-0.626	0.111
7	2.51030	2.50156	2.51104	4.44514	4.44120	4.45167	5.81386	5.79064	5.82184
	0.008	-0.340	0.037	-0.006	-0.095	0.141	-0.099	-0.498	0.038
10	2.51018	2.50375	2.51046	4.44527	4.44222	4.44815	5.81658	5.79763	5.81958
	0.003	-0.253	0.014	-0.003	-0.072	0.061	-0.052	-0.378	-0.001
Exact solution [4]	2.5101			4.44542			5.81961		

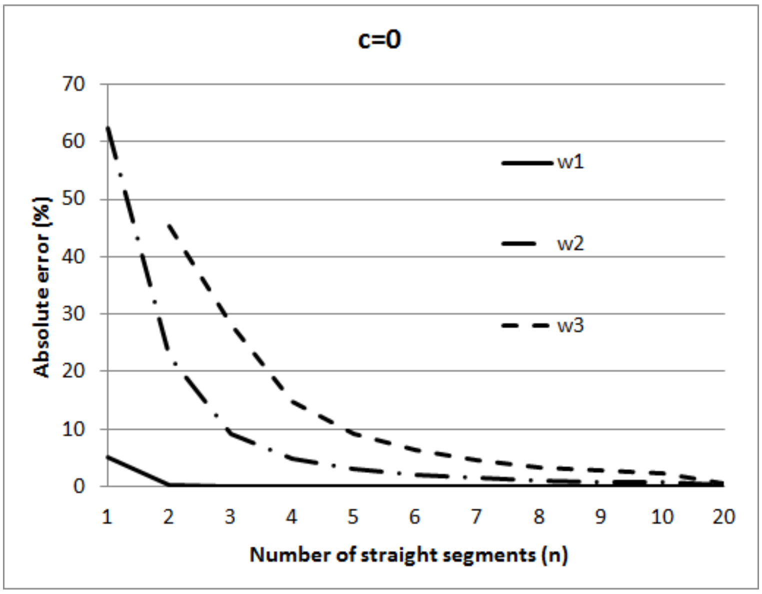

Figure 6. Absolute error of natural frequency of: straight beam (c=0) in comparison with [5]

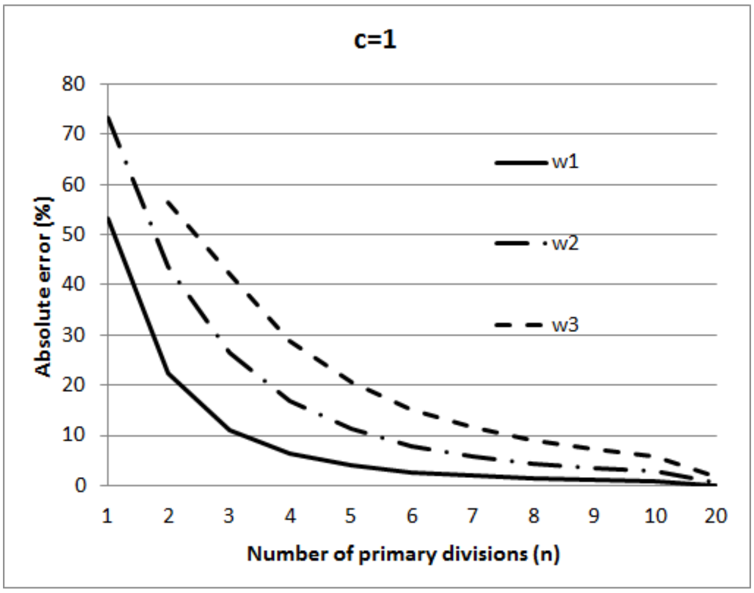

Figure 7. Absolute error of natural frequency of: maximum tapered beam (c=1), in comparison with [5]

Table 2: Natural frequencies of the cantilever beam – comparison of the present paper results and the results from [5]

c	Non-dimensional frequencies								
	$\bar{\omega}_1$			$\bar{\omega}_2$			$\bar{\omega}_3$		
	Our results		Ref. [5]	Our results		Ref. [5]	Our results		Ref. [5]
	n=10	n=20		n=10	n=20		n=10	n=20	
0	3.5169	3.5162	3.5160	21.8890	21.9984	22.035	60.3636	61.3642	61.6970
0.1	3.6307	3.6309	3.6310	22.0992	22.2156	22.254	60.5474	61.5696	61.9100
0.2	3.7612	3.7624	3.7629	22.3361	22.4607	22.502	60.7573	61.8044	62.1530
0.3	3.9125	3.9152	3.9160	22.6073	22.7417	22.786	61.0019	62.0784	62.436
0.4	4.0913	4.0956	4.0970	22.9240	23.0704	24.021	61.2942	62.4068	62.776
0.5	4.3067	4.3130	4.3152	23.3039	23.4659	23.519	61.6556	62.8145	63.199
0.6	4.5728	4.5822	4.5853	23.7773	23.9606	24.021	62.1238	63.3458	63.751
0.7	4.9134	4.9271	4.9317	24.4012	24.6162	24.687	62.7728	64.0892	64.527
0.8	5.3703	5.3907	5.3976	25.2983	25.5668	25.656	63.7744	65.2537	65.747
0.9	6.0272	6.0595	6.0704	26.7946	27.1727	27.299	65.6527	67.4945	68.115
1	7.0805	7.1374	7.1422	30.1108	30.8063	30.970	71.1455	74.3753	75.653

5. CONCLUSION

This paper presents a new method of approximative determination of frequency of the tapered cantilever beam which can serve as an alternative to relevant approaches from [1], [2] and [3]. In such discretization of the flexible tapered cantilever beam, a well-developed methodology for mechanics of a system of rigid bodies is used for the formation of the characteristic problem. It results in obtaining a computer-efficient algorithm for determination of approximate values of frequencies of the beam. Comparison of our method with the results from relevant approaches in [1], [2] and [3] was carried out on the example from paper [4]. It was shown that the relative errors of obtained frequencies in relation to the exact values given in paper [4] are smaller for first two frequencies if our approach is used than if the approaches from [1], [2] and [3] are used. For the third frequency approach from [2] give slightly better results. Then the results of our approach are compared with the results from [5]. In [5], the *Initial value method* was used for analysis of free vibration of the beam, where the *Runge-Kutta method* of numerical integration was used for determination of frequencies. That is why this algorithm is demanding in terms of computing. It was shown that the results obtained by using our approach agree to a considerable extent with the results from [5].

Based on everything previously stated, it is clear that the presented method is less demanding in terms of computing than the algorithm presented in [5], and it achieves better results than the relevant algorithms from [1], [2] and [3] for first two natural frequencies. The presented methodology can also be used for treating more complex models of flexible beams, frames, etc.

However, the question remains how the position of the prismatic joint in the rigid multibody model of the tapered cantilever beam affects the accuracy of the exposed algorithm. We assume that prismatic joint is placed in the middle of the first half of the beam. In [1] the authors have shown for which partition coefficient p algorithm achieved the best accuracy and simple approximate model of the beam. This analysis will be the subject of further research by authors.

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REFERENCES

[1] Schiehlen WO, Rauh J, “Modeling of flexible multibeam systems by rigid-elastic superelements”, Rev Bras Cienc Mec 8(2):151–163, (1986)

[2] Z.A. Parszewski, J.M.Krodkiewski, J. Skoraczynski, “RIGID FEM INA CASE HISTORY OF BOILER FEED PUMP VIBRATION”, Computers&Structures, Vol. 31, No.1, pp.103-110, (1989)

[3] J.M.Krodkiewski, “MECHANICAL VIBRATION”, THE UNIVERSITY OF MELBOURNE, Department of Mechanical and Manufacturing Engineering, (2008)

[4] S. Naguleswaran, “Vibration of an Euler–Bernoulli Beam on Elastic End Supports and with up to Three Step Changes in Cross-Section”, *International Journal of Mechanical Sciences*, **44**, pp. 2541-2555, (2002)

[5] C.Y. Wang, “Vibration of a tapered cantilever of constant thickness and linearly tapered width”, Arch. Appl. Mech. Vol. 83, pp. 171-176, (2013)

[6] S. Šalinić, “Modeling of a light elastic beam by a system of rigid bodies”, Theoret. Appl. Mech., Vol. 31, No. 3-4, pp. 395-410, (2004)

[7] S. Šalinić, A. Nikolić, “On the determination of natural frequencies of a cantilever beam in free bending vibration: a rigid multibody approach”, Forsch Ingenieurwes, Vol.77: pp.95–104, (2013)

[8] L. Meirovitch, “Fundamentals of vibrations”, McGraw-Hill, New York (2001)

[9] V. Čović, M. Lazarević, “Mechanics of robots” Faculty of Mechanical Engineering, University of Belgrade (in Serbian) (2009)